



PERGAMON

Journal of Quantitative Spectroscopy &
Radiative Transfer 78 (2003) 85–98

Journal of
Quantitative
Spectroscopy &
Radiative
Transfer

www.elsevier.com/locate/jqsrt

Radiative transfer through inhomogeneous turbid media: implementation of the adjoint perturbation approach at the first order

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Received 15 May 2002; accepted 9 August 2002

Abstract

Recently, the advantages of application of the perturbation technique which is based on joint use of both the direct and adjoint solutions of the radiative transfer equation to solve and analyze some 1D problems of atmospheric physics has been demonstrated. In this paper this technique is applied to problems of radiative transfer through spatially inhomogeneous scattering and absorbing media. This technique is shown to allow one both to obtain the solution with reasonable accuracy and to get physical insight into the problem under consideration. The accuracy of the perturbation technique is demonstrated through comparison with results from the SHDOM simulation code for one problem of cloud optics.

Published by Elsevier Science Ltd.

Keywords: 3D radiative transfer; Cloud optics; Perturbation technique

1. Introduction

The problem of radiative transfer through an inhomogeneous three-dimensional scattering medium has a long history and can also be easily traced to corresponding problems in neutron reactor theory [1,2] (for example, the calculation of the critical mass), so that most modern approaches to describe radiation propagation [3,4] have a corresponding analog in that theory. Fortunately, the requirements

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on the numerical precision of the estimation of radiation effects, and of the radiance characteristics, are not as high as, say, for the critical mass calculation (where a small overestimation might lead to a catastrophe). Here the possibility to get insight into the nature of the particular problem and/or to evaluate at the same time the effect of different variations of the medium parameters on the radiation field characteristics with reasonable accuracy, are sometimes more important than the ability to obtain an additional “true” digit after the decimal point.

In fact, the modern level of knowledge of the three-dimensional structure of typical media (clouds in the theory of climate [5], tissue in medical imaging [6]) does not allow one to build optical models of the corresponding scattering media without introducing some simplification and approximation, and this undoubtedly leads to the accuracy of the prediction being limited by such lack of information. This is one of the basic reasons why different approximations (for example, diffusion) are widely and successfully used to describe the phenomena of radiance propagation both in the theory of medical imaging and image reconstruction [7], and the study of the radiative effects of three-dimensional structure of real clouds [8]. Despite the fact that in some situations an analytical solution in the diffusion approximation can be obtained as it was done by Davis and Marshak for the cloud albedo [8], even such a simplification assumes the necessity to solve a second-order differential equation with non-constant coefficients, the solution of which is not always that simple and obvious.

An important simplification can be achieved if the horizontal inhomogeneity is not strong with respect to the average characteristics of the medium. This assumption allows one to obtain a simpler solution using a special numerical technique (for example, the gradient correction method [9]) or to develop the perturbation approach [10] on the basis of introducing a small parameter into the radiative transfer equation. Such a technique within the framework of the diffusion approximation was used by Li et al. [11] to investigate the effect of a small sine-shape variation of the extinction coefficient on the characteristics of the upwelling and downwelling radiation. However, as a rule the variation of only one parameter, such as the extinction coefficient, was considered, and it was necessary to reconsider the problem from the very beginning if variation of another medium characteristic has to be considered. This drawback is a consequence of the particular perturbation technique that was used.

A different type of perturbation approach may be formulated on the basis of the joint consideration of the direct and adjoint formulation of the same problem [12–14]. The most prominent advantage of this approach is that it allows one to study the effect of perturbations of different natures (for example, phase function or single scattering albedo variations) on the radiance characteristics. The main goal of this paper is to provide general equations relevant to such a perturbation calculation, which can be used in the optics of horizontally inhomogeneous scattering media. Additionally, we will demonstrate the ability of this technique, which can make qualitative analysis simpler and more efficient, by consideration of selected problems of atmospheric optics, and validate it through comparison with the results of numerical simulation of the same problem.

The paper’s organization is as follows. Section 2 introduces the basic notation and equations, which describe both the direct and adjoint radiative transfer equations with the appropriate boundary conditions. The basic equation of the perturbation approach for the horizontally inhomogeneous scattering medium can be found in Section 3. Section 4 is devoted to comparison of the perturbation theory prediction with an SHDOM simulation of the same problem.

2. Direct and adjoint formulations of the radiative transfer problem

2.1. Direct formulation

Let us consider radiance propagation through a scattering and/or absorbing medium bounded by a convex surface $S(\vec{r})$. In this case the equation of radiative transfer [15] may be written as

$$[\vec{n} \cdot \vec{\nabla} + \sigma_e(\vec{r})]I(\vec{r}, \vec{n}) = \sigma_s(\vec{r}) \int_{4\pi} P(\vec{r}, \vec{n}' \rightarrow \vec{n})I(\vec{r}, \vec{n}') d\vec{n}' + Q(\vec{r}, \vec{n}). \quad (1)$$

Here, $I(\vec{r}, \vec{n})$ is the specific radiance at the point \vec{r} in the direction \vec{n} , $\vec{\nabla}$ is the gradient with respect to \vec{r} , $\sigma_e(\vec{r})$ and $\sigma_s(\vec{r})$ are the extinction and scattering coefficients at the point \vec{r} , respectively, $P(\vec{r}, \vec{n}' \rightarrow \vec{n})$ is the phase function, and $Q(\vec{r}, \vec{n})$ represents the sources of radiation, such as a laser beam, solar radiation or internal sources of emission. The phase function is normalized as follows:

$$\int_{4\pi} P(\vec{r}, \vec{n}' \rightarrow \vec{n}) d\vec{n} = 1. \quad (2)$$

The boundary condition is formulated on the principle “no radiance comes into the scattering medium from outside”. In the case of a reflective boundary surface it has the form

$$I(\vec{r}, \vec{n}) = \frac{1}{\pi} \int_{\vec{n}' \cdot \vec{n}_\perp = \mu' < 0} M(\vec{r}, \vec{n}, \vec{n}')I(\vec{r}, \vec{n}')|\mu'| d\vec{n}' \quad \text{for } \vec{r} \in S(\vec{r}) \text{ and } \vec{n} \cdot \vec{n}_\perp = \mu > 0, \quad (3)$$

where \vec{n}_\perp is the normal to the boundary surface, which is directed into the medium, and $M(\vec{r}, \vec{n}, \vec{n}')$ is the function which describes the reflection properties of the boundary surface. In the simplest case of Lambertian reflection, $M(\vec{r}, \vec{n}, \vec{n}') = M(\vec{r}) \leq 1$. In order to simplify the necessary manipulation it is more convenient to use an operator form of notation. Eq. (1) then takes the form

$$\hat{\mathbf{L}}I(\vec{r}, \vec{n}) = Q(\vec{r}, \vec{n}) \quad (4)$$

and the transport operator $\hat{\mathbf{L}}$ is obviously defined as

$$\hat{\mathbf{L}} = \vec{n} \cdot \vec{\nabla} + \sigma_e(\vec{r}) - \sigma_s \int_{4\pi} d\vec{n}' P(\vec{r}, \vec{n}' \rightarrow \vec{n}) \circ. \quad (5)$$

(Note that the notation \circ is used to indicate that the final term is an integral operator, not merely a definite integral.)

2.2. Adjoint formulation

Let the scalar product of two functions $g(\vec{r}, \vec{n})$ and $h(\vec{r}, \vec{n})$ be introduced as integration over the entire range Ξ of the problem variables \vec{r}, \vec{n}

$$\langle g, h \rangle = \int_{\Xi} \int g(\vec{r}, \vec{n})h(\vec{r}, \vec{n}) d\vec{r} d\vec{n}. \quad (6)$$

Note that the characteristics of the radiance field in the medium, or the result of any optical measurements, may be written using this scalar product of $I(\vec{r}, \vec{n})$ and some receiver function $R(\vec{r}, \vec{n})$:

$$E = \int_{\Xi} \int I(\vec{r}, \vec{n})R(\vec{r}, \vec{n}) d\vec{r} d\vec{n}. \quad (7)$$

For example, if we calculate the vertical downward flux $F(\vec{r}_0) = \int_{2\pi} I(\vec{r}_0, \vec{n}) \mu d\vec{n}$ at the point \vec{r}_0 , the receiver function has the form $R(\vec{r}, \vec{n}) = \mu \eta(\mu) \delta(\vec{r} - \vec{r}_0)$ ($\eta(x)$ is the unit step function); for the case of a radiance measurement at the point \vec{r}_0 in the direction \vec{n}_0 , $R(\vec{r}, \vec{n}) = \delta(\vec{r} - \vec{r}_0) \delta(\vec{n} - \vec{n}_0)$.

Let us define the adjoint radiative transfer problem as

$$\hat{\mathbf{L}} \tilde{I}(\vec{r}, \vec{n}) = R(\vec{r}, \vec{n}), \quad (8)$$

where $R(\vec{r}, \vec{n})$ is the receiver function of the direct problem and $\hat{\mathbf{L}}$ is the adjoint transfer operator, which obeys the functional relation

$$\langle \tilde{I}, \hat{\mathbf{L}} I \rangle = \langle \hat{\mathbf{L}} \tilde{I}, I \rangle. \quad (9)$$

Following a procedure similar to that described by Box et al. [14], one can deduce the explicit forms of both the adjoint operator

$$\hat{\mathbf{L}} = -\vec{n} \cdot \vec{\nabla} + \sigma_e(\vec{r}) - \sigma_s(\vec{r}) \int_{4\pi} d\vec{n}' P(\vec{r}, -\vec{n}' \rightarrow -\vec{n}) \circ \quad (10)$$

and the boundary condition, which the adjoint intensity $\tilde{I}(\vec{r}, \vec{n})$ is subject to, namely

$$\tilde{I}(\vec{r}, \vec{n}) = \frac{1}{\pi} \int_{\vec{n}' \cdot \vec{n}_\perp = \mu' > 0} M(\vec{r}, -\vec{n}', -\vec{n}) \tilde{I}(\vec{r}, \vec{n}') \mu' d\vec{n}' \quad \text{for } \vec{r} \in S(\vec{r}) \text{ and } \vec{n} \cdot \vec{n}_\perp = \mu < 0 \quad (11)$$

for a reflective boundary, and “no exiting adjoint photons” otherwise. Comparing (3) and (5) with (10) and (11) we notice that if $R(\vec{r}, \vec{n}) = Q(\vec{r}, -\vec{n})$ then

$$\tilde{I}(\vec{r}, \vec{n}) = I(\vec{r}, -\vec{n}) \quad (12)$$

and the solution of a given adjoint problem can be obtained through solving the corresponding direct problem. This means that all the results which can be obtained in the direct formulation and all the mathematical and numerical techniques developed to solve the direct problem may be equally obtained and used in the adjoint one and vice versa. The particular method of solution can be determined from the point of view of calculational convenience or even of personal preference. More discussion of this point can be found in the paper by Box et al. [16].

2.3. Perturbation technique

As is discussed by Marchuk [12] and Box et al. [14], if we have the solution for both formulations of a given problem at the same time, we can approximately predict the effect of a small variation, $\delta \hat{\mathbf{L}}$, of the transfer operator $\hat{\mathbf{L}}$, on the result of calculation of the effect E by means of the simple formula

$$\delta E \approx -\langle \tilde{I}, \delta \hat{\mathbf{L}} I \rangle. \quad (13)$$

On the face of it, it seems that we have come to the necessity to solve a more complicated problem than the initial one, so that we need to solve two problems instead of one. However, this impression does not survive a closer scrutiny. For example, let us consider the problem of radiation propagation through a three-dimensional inhomogeneous medium, which can be considered as infinite in the horizontal directions. Unbroken cloud fields in atmospheric optics, and tissue in medical imaging,

are good examples of such media. A natural simplification follows immediately after we represent the operator $\hat{\mathbf{L}}$ as the sum

$$\hat{\mathbf{L}} = \hat{\mathbf{L}}_b + \delta\hat{\mathbf{L}}, \tag{14}$$

where

$$\hat{\mathbf{L}}_b = \lim_{D \rightarrow \infty} \frac{\int_D \hat{\mathbf{L}} d\vec{r}_\perp}{\int_D d\vec{r}_\perp}, \quad \delta\hat{\mathbf{L}} = \hat{\mathbf{L}} - \hat{\mathbf{L}}_b. \tag{15}$$

Here $\vec{r}_\perp = (x, y) \in D$, and D denotes the area of admissible values of x and y . In this case, we need to solve only the problem of the radiance propagation through a stratified horizontally homogeneous medium, which is substantially simpler [3,4]. Then, presuming the corresponding adjoint problem solved, Eq. (13) provides us immediately with an estimation of how small variations of the medium characteristics, which are arbitrary in their nature, affect a given radiance characteristic or radiative effect under investigation. Note that Eq. (13) accurately describes the first order effect only, which is reasonably good for a number of problems. Despite its simplicity, this equation allows one immediately to make some qualitative conclusions based on the symmetry of the problem.

For example, let us consider the problem of solar illumination of an unbroken cloud field, and consider the effect of weak horizontal inhomogeneity on the domain average characteristics like average cloud albedo. In this case, the corresponding receiver function is horizontally homogeneous, $R(\vec{r}, \vec{n}) = R(z, \vec{n})$. It follows immediately from (13) that

$$\begin{aligned} \delta E &= - \int \int \tilde{I}(z, \vec{n}) \delta\hat{\mathbf{L}} I(z, \vec{n}) d\vec{r} d\vec{n} \\ &= - \int \tilde{I}(z, \vec{n}) \left[\int \delta\hat{\mathbf{L}} d\vec{r}_\perp \right] I(z, \vec{n}) d\vec{n} dz \end{aligned} \tag{16}$$

and since $\int \delta\hat{\mathbf{L}} d\vec{r}_\perp = 0$, then $\delta E = 0$ which means that such characteristics are less sensitive to small variation, and to calculate them accurately it is necessarily to perform a more detailed calculation, at least to take into account terms of the second order as was done by Li et al. [17] and Box et al. [18].

3. Implementation of the perturbation technique at first order

To validate the proposed approach let us consider the problem of how a weak sinusoidal inhomogeneity in the extinction coefficient affects the results of satellite radiance observations [11]. Let the Sun illuminate a plane parallel cloud with only horizontal variation of the extinction coefficient:

$$\sigma_c(\vec{r}) = \bar{\sigma}_c [1 - \varepsilon \sin(k_0 x)]. \tag{17}$$

Here $\bar{\sigma}_c$ is the average extinction coefficient, ε is the relative amplitude of the perturbation, and k_0 is the wavenumber which describes the characteristic scale of the extinction coefficient variation. The single scattering albedo ω_0 and the phase function are assumed to be the same at all points inside

the cloud. The source and receiver functions may be written in the form

$$Q(\vec{r}, \vec{n}) = \mu_0 \delta(z) \delta(\vec{n} - \vec{n}_s), \quad (18a)$$

$$R(\vec{r}, \vec{n}) = \delta(\vec{r} - \vec{r}_r) \delta(\vec{n} - \vec{n}_r). \quad (18b)$$

Here \vec{n}_s and \vec{n}_r are the directions of the Sun illumination and observation, respectively, μ_0 is the cosine of the solar zenith angle, and \vec{r}_r is the position of the receiver.

Following (14) we subdivide our medium into the base medium, with characteristics which are the average of the corresponding parameters of the cloud, plus the perturbation. The radiative transfer operator $\hat{\mathbf{L}}$ for the base medium and the perturbation $\delta\hat{\mathbf{L}}$ can be written as

$$\hat{\mathbf{L}} = \vec{n} \cdot \vec{\nabla} + \bar{\sigma}_e \left[1 - \omega_0 \int_{4\pi} d\vec{n}' P(\vec{n}' \rightarrow \vec{n}) \circ \right], \quad (19a)$$

$$\delta\hat{\mathbf{L}} = -\bar{\sigma}_e \varepsilon \sin(k_0 x) \left[1 - \omega_0 \int_{4\pi} d\vec{n}' P(\vec{n}' \rightarrow \vec{n}) \circ \right]. \quad (19b)$$

However, to perform the perturbation calculation using (13) we need the explicit forms of both $I(\vec{r}, \vec{n})$ and $\tilde{I}(\vec{r}, \vec{n})$ for all possible values of (\vec{r}, \vec{n}) . We shall discuss only the solution in the direct formulation since as mentioned above, the same technique can be used to solve the adjoint problem.

Having assumed the necessity to perform an integration over angles in (13), it is convenient to use the spherical harmonics or P_N approximation [3,19,20], which can be obtained by expansion of $I(\vec{r}, \vec{n})$ into spherical harmonics and assuming that only a limited number of expansion coefficients are significantly different from zero. In this case $I(\vec{r}, \vec{n})$ has the form

$$I(\vec{r}, \vec{n}) = \frac{1}{4\pi} \sum_{m=-M}^M \sum_{n=m}^{N_m} (2n+1) P_n^m(\mu) \psi_n^m(\vec{r}) e^{im\phi}, \quad (20)$$

where M and N_m are parameters which determine the order of the approximation, and hence its accuracy. As discussed in the book by Lenoble [3] there is no explicit rule to estimate how large M and N_m should be to meet a given accuracy requirement, and only numerical experimentation can help to estimate them properly. To finish this short introduction into the spherical harmonics approximation it should be added that representation (20) does not satisfy the boundary condition (3) exactly at every \vec{n} , and that is why either the Marshak [21] or Mark [22] forms of discretization are usually used. The discussion about their advantages can be found in Refs. [1,19,23].

A further simplification comes from the particular shape of the inhomogeneity chosen. Note that, because of our base medium's symmetry, it is clear that the adjoint intensity $\tilde{I}(\vec{r}, \vec{n})$ depends on the spatial variable $\vec{r}_\perp = (x, y)$ in the form

$$\tilde{I}(\vec{r}, \vec{n}) = \tilde{I}(z, \vec{r}_\perp - \vec{r}_{\perp r}, \vec{n}), \quad (21)$$

where $\vec{r}_{\perp r}$ is the horizontal position of the receiver. This means that the effect variation $\delta E(\vec{r}_{\perp})$ can be written as

$$\delta E(\vec{r}_{\perp}) = - \int_{\Xi} \int_{\Xi} \int \tilde{I}(z, \vec{r}_\perp - \vec{r}_{\perp r}, \vec{n}) \delta\hat{\mathbf{L}}(\vec{r}_\perp) I(z, \vec{n}) d\vec{r}_\perp dz d\vec{n}. \quad (22)$$

In this situation it is simpler to consider not the functions $I(\vec{r}, \vec{n})$ and $\delta E(\vec{r}_\perp)$, but their Fourier transforms with respect to \vec{r}_\perp :

$$I(z, \vec{k}, \vec{n}) = \int_{-\infty}^{\infty} \exp(-i\vec{k}\vec{r}_\perp) I(\vec{r}_\perp, \vec{n}) d\vec{r}_\perp, \quad (23a)$$

$$E(\vec{k}) = \int_{-\infty}^{\infty} \exp(-i\vec{k}\vec{r}_\perp) E(\vec{r}_\perp) d\vec{r}_\perp. \quad (23b)$$

We can now rewrite formula (13) for the effect using Fourier transforms of the corresponding function and operator, and the convolution theorem, to obtain

$$\delta E(\vec{k}) = - \int_{\Xi} \int \tilde{I}(z, \vec{k}, \vec{n}) \delta \hat{\mathbf{L}}(\vec{k}) I(z, \vec{n}) d\vec{n} dz. \quad (24)$$

Note also that in Fourier space the transfer operator $\hat{\mathbf{L}}_b$ for the base case, and its variation $\delta \hat{\mathbf{L}}$, have the forms

$$\hat{\mathbf{L}}_b(\vec{k}) = \mu \frac{d}{dz} - i\vec{n}_\perp \cdot \vec{k} + \bar{\sigma}_e \left[1 - \omega_0 \int_{4\pi} d\vec{n}' P(\vec{n}' \rightarrow \vec{n}) \circ \right], \quad (25a)$$

$$\delta \hat{\mathbf{L}}(\vec{k}) = 2\pi^2 i \varepsilon \bar{\sigma}_e [\delta(\vec{k} - \vec{k}_0) - \delta(\vec{k} + \vec{k}_0)] \left[1 - \omega_0 \int_{4\pi} d\vec{n}' P(\vec{n}' \rightarrow \vec{n}) \circ \right]. \quad (25b)$$

Comparing this form of $\hat{\mathbf{L}}$ with the corresponding form of the transfer operator for the one-dimensional problem [15], it may be seen that they are formally equivalent if we consider $(i\vec{n}_\perp \cdot \vec{k})$ as an addition to the extinction coefficient (in fact, an imaginary absorption). This means we have managed to simplify the initial problem substantially. To conclude discussion about the computational technique used, it should be noted that the spherical harmonics approximation allows all integrals of the radiance over angles to be estimated with high accuracy, but unavoidable oscillations in the radiance angle distribution appear near the boundary. To improve the situation an iteration procedure [3] based on using the formal solution of the radiative transfer equation may be used. The additional advantage of this procedure is that smaller M and N_m , which characterize the number of terms taken into account in the expansion (20), may be used to obtain a sufficiently accurate solution. We will refer to it as the “improved” spherical harmonics approximation.

4. Comparison with the SHDOM

How accurate is our technique? To answer this question we performed a simulation of the effect of a small modulation of the extinction coefficient on satellite observation using two techniques: the perturbation approach and SHDOM code [24], which is freely distributed [25]. We considered a slab of scattering medium, which had geometrical thickness, H , and was characterized by the single scattering albedo $\omega_0 = 1.0$ and Henyey–Greenstein phase function [26], defined as

$$P(\cos \beta) = \frac{1}{4\bar{u}} \frac{1 - g^2}{[1 + g^2 - 2g \cos(\beta)]^{3/2}}. \quad (26)$$

Here β is the scattering angle, and g is the asymmetry parameter of the phase function.

The base medium is a homogeneous slab characterized by the extinction coefficient $\bar{\sigma}_e$, which provided an average optical thickness of $\bar{\sigma}_e H = 10.0$. The extinction coefficient dependence of the

perturbed medium in the form (17) has the same $\bar{\sigma}_e$, with $k_0 = 0.1\bar{\sigma}_e$ and $\varepsilon = 0.1$. The optical length of the extinction coefficient oscillation period is $2\pi\bar{\sigma}_e/k_0 = 20\pi \approx 62.8$.

For this problem as follows from (23)–(25) the effect variation, δE , in real space has the form

$$\delta E(\vec{r}_\perp) = \frac{\varepsilon}{2i} [\Psi(\vec{k}_0) \exp(i\vec{k}_0 \cdot \vec{r}_\perp) - \Psi(-\vec{k}_0) \exp(-i\vec{k}_0 \cdot \vec{r}_\perp)], \quad (27)$$

where

$$\Psi(\vec{k}) = \int \int \tilde{I}(z, \vec{k}, \vec{n}) \left[I(z, \vec{n}) - \omega_0 \int_{4\pi} P(\vec{n}, \vec{n}') I(z, \vec{n}') d\vec{n}' \right] dz d\vec{n}. \quad (28)$$

Taking into account that $\tilde{I}(z, \vec{r}_\perp, \vec{n})$ is real and, hence,

$$\tilde{I}^*(z, \vec{k}, \vec{n}) = \tilde{I}(z, -\vec{k}, \vec{n}), \quad (29)$$

where (*) denotes an operation of complex conjugation, it is clear that

$$\Psi^*(\vec{k}) = \Psi(-\vec{k}). \quad (30)$$

Representing $\Psi(\vec{k})$ as

$$\Psi(\vec{k}) = \rho(\vec{k}) \exp[i\Delta\phi(\vec{k})], \quad (31)$$

where $\rho(\vec{k}) = |\Psi(\vec{k})|$ and $\Delta\phi(\vec{k}) = \arg[\Psi(\vec{k})]$, we obtain immediately that

$$\delta E(\vec{r}_\perp) = \varepsilon \rho(\vec{k}_0) \sin[\vec{k}_0 \vec{r}_\perp + \Delta\phi(\vec{k}_0)]. \quad (32)$$

To perform the SHDOM simulation a spatial grid of 120×50 in the x – z plane, and an angle resolution of 16 streams with 32 azimuthal modes was used (in SHDOM terminology [24]). The improved spherical harmonics approximation was used to solve both the direct and adjoint problems for the base medium during the perturbation technique calculation. It was found that $M = 3$ and $N_m = 7$ are enough to achieve an accuracy $\leq 0.5\%$ in the estimation of the corresponding radiance for our particular problem when $g \leq 0.5$.

The results of the perturbation calculation (solid lines) and SHDOM simulation (depicted by the marks) are shown in Fig. 1 for $g = 0$ and Fig. 2 for $g = 0.5$, for a solar zenith cosine μ_0 of 1.0 (labeled 1) and 0.3 (labeled 2). We see that the phase of the oscillation of the observed upwelling radiation along the x -axis for $\mu_0 = 0.3$ does not coincide with the phase of the extinction coefficient modulation, while it does for $\mu_0 = 1.0$, by symmetry. We call this phenomenon a “shift” following Li et al. [11]. The reason for the shift is clear and follows immediately from the low order scattering consideration of the optimal route which the light has to follow to produce the maximum upwelling radiance.

Comparison of the perturbation technique prediction with the outcome of the SHDOM simulation shows that they coincide reasonably well. As seen from these two figures, the perturbation technique provides an accurate estimation of both the amplitude of the oscillation and the shift. However, there is some deviation between the perturbation technique and the SHDOM results (the asymmetry of the real oscillation with respect to the zero line; the real shape of the oscillation differs slightly from the form of a perfect sine curve), which appears because the perturbation technique takes into account only the effect of the first order term with respect to $\delta\hat{\mathbf{L}}$. To describe such an effect at least the second or possibly higher order terms should be incorporated as was done by Li et al. [17] in the diffusion approximation. The full formalism for higher order terms for the general radiative transfer theory is developed in the accompanying paper by Box et al. [18].

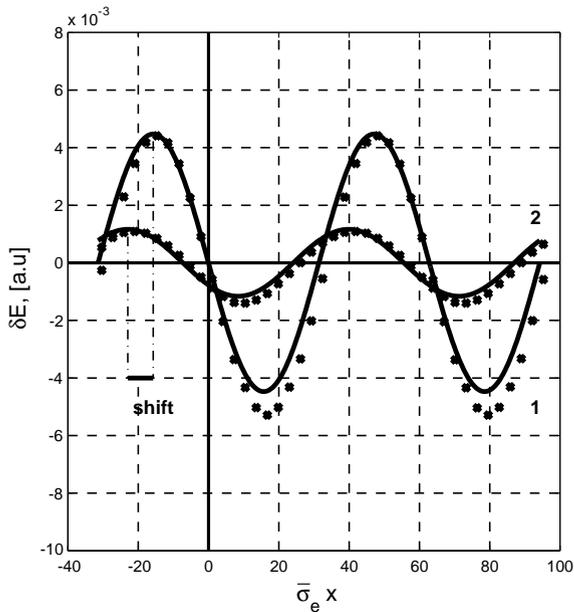


Fig. 1. The effect variation δE as a function of $\bar{\sigma}_e x$ calculated using the perturbation approach (solid line) and the SHDOM (symbols). The average cosine of the phase function, g , is 0. The cosine μ_0 of the solar zenith angle is 1.0 (1) and 0.3 (2).

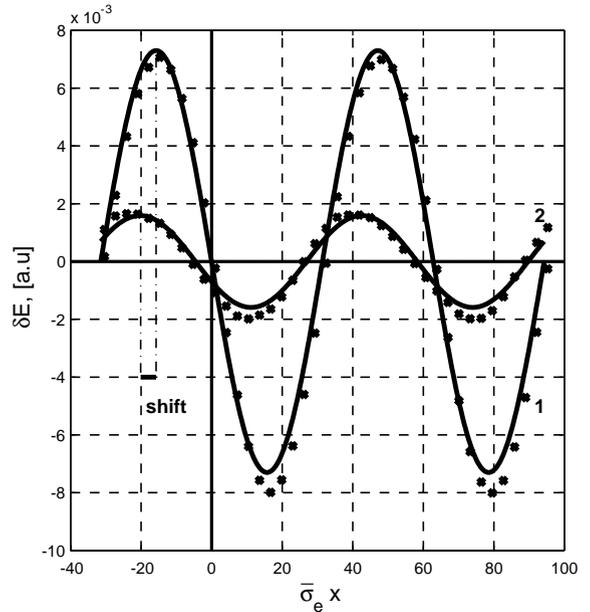


Fig. 2. The same as Fig. 1, but $g = 0.5$.

Within the framework of the perturbation approach, as follows from Eq. (32), the shift can be estimated just as $\Delta\phi(\vec{k}_0) = \arg[\Psi(\vec{k}_0)]$, and it does not depend on the relative amplitude of the extinction coefficient variation, ε . We emphasize once more that our perturbation consideration can describe only the first-order effects which may be insufficient to describe all parameters with high enough accuracy, and significant error in some characteristics may appear. To make the situation clearer let us study how the “shift” depends on the phase function asymmetry parameter. The simulation was made for the same cloud model as above for three values of g : 0.0, 0.5 and 0.85. Fig. 3 shows the results of the perturbation technique (depicted by the solid lines) and the estimation based on the SHDOM simulation data (marks), where the “shift” is calculated as the offset between the horizontal positions of the upwelling radiance maximum for the slanted and normal illumination. We see that the accuracy of the perturbation approach prediction decreases with an increase of the asymmetry parameter of the phase function.

What is the reason for such a disagreement? To understand it let us analyze Eq. (13) taking into account (19)

$$\delta E = \varepsilon \bar{\sigma}_e \left[- \int_{\Xi} \int \tilde{I}(\vec{r}, \vec{n}) \sin(k_0 x) I(z, \vec{n}) d\vec{r} d\vec{n} + \omega_0 \int_{\Xi} \int \tilde{I}(\vec{r}, \vec{n}) \sin(k_0 x) \int_{4\pi} P(\vec{n}, \vec{n}') I(z, \vec{n}) d\vec{n}' d\vec{r} d\vec{n} \right]. \quad (33)$$

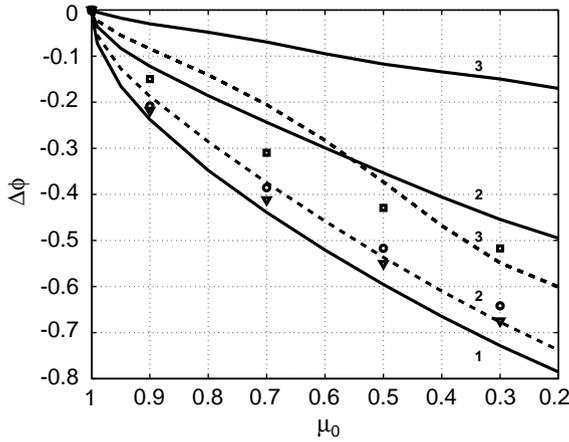


Fig. 3. The “shift” of the maximum of the upwelling radiance as a function of the cosine μ_0 of the solar zenith angle calculated using the perturbation approach (solid line), the SHDOM (marks), and the perturbation approach with the redefined phase function (dashed line). The average cosine of the phase function, g , is 0.0 (1, triangles), 0.5 (2, circles), and 0.85 (3, squares).

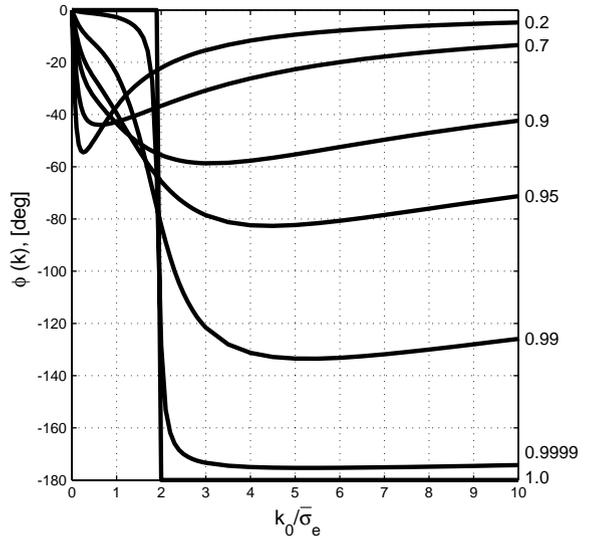


Fig. 4. The “shift” of the maximum of the upwelling radiance as a function of $k_0/\bar{\sigma}_e$. The average cosine of the phase function, g , is 0.0. The figures at the curves show the solar zenith cosine μ_0 .

This equation explicitly suggests a model of the phenomenon considered: the variation of the effect, δE , can be represented as a sum of independent contributions from all optical depths. One of the consequences of such a model is that the effect of the extinction coefficient variation is overestimated. This leads to an underestimation of the “shift”. To improve the situation we should include even approximately the interaction between neighboring layers. The simplest approach follows from the fact that light scattered through small angles behaves similar to unscattered light: it propagates in approximately the same direction, and hence should be treated as unscattered light rather than being removed. Such a problem can be addressed by a redefinition of the phase function as is done within the δ -Eddington approximation [3]

$$P(\cos \beta) = \frac{1}{4\bar{u}}g\delta(1 - \cos \beta) + (1 - g). \tag{34}$$

Now we may expect that the quality of the perturbation prediction will improve. The numerical simulation results (shown in Fig. 3 by dashed lines) confirm our expectation. We should point out that the effective medium is of quite common use to obtain a physically adequate model adjusted to study some particular phenomenon [4,8,20].

To complete the study of the “shift” let us consider how it depends on the wavenumber, k_0 . We performed calculations for our model medium with a phase function asymmetry parameter of 0.0. Fig. 4 shows the results of our simulation for several solar zenith cosines, the values of which are depicted by the numbers beside the curves. As can be seen from in this figure, at small $k_0 < 0.25\bar{\sigma}_e$, the smaller μ_0 the greater the “shift”, but when $k_0 > 2.0\bar{\sigma}_e$ the dependence becomes opposite in character. This

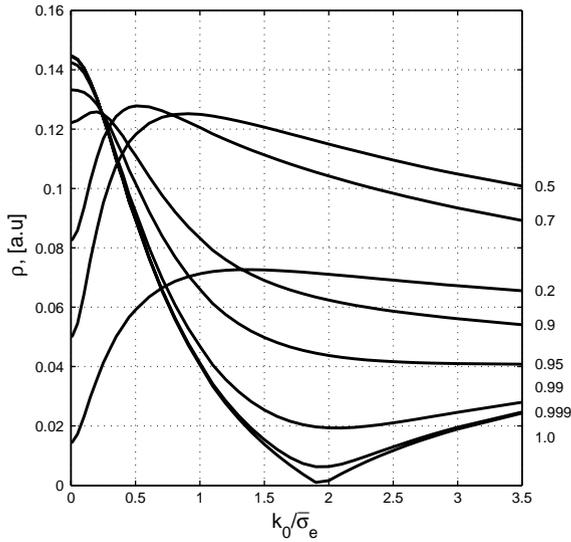


Fig. 5. Amplitude coefficient ρ as a function of $k_0/\bar{\sigma}_e$. The average cosine of the phase function, g , is 0.0. The figures at the curves show the solar zenith cosine μ_0 .

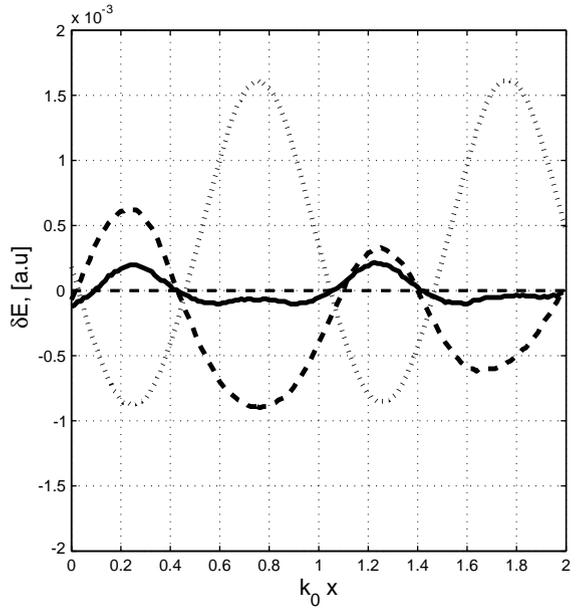


Fig. 6. The upwelling zenith radiance profile as a function of k_0x simulated using the SHDOM. The average cosine of the phase function, g , is 0.0. The wavenumber k_0 is $1.0\bar{\sigma}_e$ (dotted line), $2.0\bar{\sigma}_e$ (solid line), and $3.0\bar{\sigma}_e$ (dashed line).

means that the less optically dense regions reflect more radiance than more dense regions. This phenomenon was described by Li et al. [11] as correlated and anticorrelated distributions. We can obtain additional information by analyzing how $\rho(\vec{k}_0)$ depends on \vec{k}_0 , which is depicted in Fig. 5. The figure shows that when $k_0 \approx 2.0\bar{\sigma}_e$ and the direction of Sun illumination is close to the normal, the amplitude coefficient, $\rho(\vec{k}_0)$, tends to be quite small, and hence, the expected variation of the measured upwelling radiance should also be small and as a result the satellite measurements may not allow one to distinguish such a perturbed medium from a homogeneous one. Note that this phenomenon can be observed only when the solar angle cosine $\mu_0 \geq 0.999$. To check this prediction of the perturbation approach we performed SHDOM simulations for the case $\mu_0 = 1.0$; the results are shown in Fig. 6: $k_0 = 1.0\bar{\sigma}_e$ (dotted line), $k_0 = 2.0\bar{\sigma}_e$ (solid line), and $k_0 = 3.0\bar{\sigma}_e$ (dashed line). From the figure we see that for $k_0 = 2.0\bar{\sigma}_e$ the upwelling radiance profile has substantially less oscillation amplitude and does not resemble a sine shape at all.

To understand why the anticorrelated distribution appears is simpler by considering a modification of our sunlight propagation problem. The formal solution of the radiative transfer equation [3] makes it possible to think that at every depth z there is a plane source with an angular pattern $I(z, \vec{n})$ and the solution of this problem is exactly equivalent to our sunlight propagation problem. It is clear that each source contributes to the upwelling radiance, and this contribution is a mixture of the direct and scattered radiance. The scattered radiation travels through a different area following a random path before it reaches the receiver. If the characteristic length of the extinction coefficient variation is small enough, then the scattered radiance distribution gets averaged and becomes homogeneous with respect

to the horizontal coordinate, whereas the direct radiance travels directly into the direction of the receiver and is attenuated along this path depending on the extinction coefficient at a given horizontal coordinate. It is clear that the less the extinction coefficient the greater the relative contribution of the direct radiance and an anticorrelated distribution appears.

Fig. 6 also provides us with an estimation of when the independent pixel approximation [27] (IPA), which is based on the solution of a one-dimensional problem, can be used. If the assumptions which the IPA is based on are applied to the derivation of Eq. (32), we may deduce that the effect variation in the IPA has the form

$$\delta E_{\text{IPA}}(\vec{r}_{\perp r}) = \varepsilon \rho(\vec{k}_0 = 0) \sin[\vec{k}_0 \vec{r}_{\perp}]. \quad (35)$$

Direct comparison with Eq. (32) shows that the IPA can be used when $\rho(\vec{k}_0 = 0) \approx \rho(\vec{k})$ (we are not discussing the applicability of the IPA with respect to the “shift” as this phenomenon cannot be described at all within the framework of the IPA). An examination of Fig. 6 shows that the IPA can be used with an accuracy $\approx 10\%$ if $k_0 < 0.2\bar{\sigma}_e$. However, this estimation is quite rough as the more accurate answer obviously depends on the sun angle.

To finalize our discussion, it is important to consider the question: how strong a variation of the extinction coefficient can be studied using the perturbation technique. Unfortunately, the answer cannot be provided consistently within the framework of the method without performing simulations and then we can estimate the validity of the result obtained through comparison of the perturbation term with the base problem solution. If their contributions are substantially different we may expect that the perturbation technique has managed to provide an accurate solution for that particular problem. However, it is more interesting to validate the perturbation technique using more accurate simulation methods. To do this we should first find a criterion for applicability of the perturbation technique. Considering Eq. (32) we may deduce that the ratio $\delta E(\vec{r}_{\perp})/\varepsilon$ does not depend on the extinction coefficient variation strength ε . This result is a consequence of our first-order consideration and can be used as the criterion that we are looking for. We calculated this ratio using the SHDOM for $k_0 = 1.0\bar{\sigma}_e$, $g = 0.0$, and several values of sun angle cosine μ_0 : 1.0, 0.9 and 0.5. Fig. 7 shows that even for comparatively large variation strength, $\varepsilon = 0.5$, and oblique sun illumination, the shape deviation is not pronounced and according to our criterion the perturbation technique may be successfully applied and used.

The case of normal sunlight incidence needs to be considered separately. From the figure we see that although the lines have approximately the same shape, the average level of these lines goes down with increasing ε . As follows from our criterion to describe this effect at least the second order perturbation calculation should be incorporated. However, the difference between lines which corresponding to $\varepsilon = 0.1$ and 0.2 is not large and can be neglected in many problems.

5. Conclusion

In the present paper we have implemented the first-order perturbation approach, based on joint use of the adjoint and direct solutions. The most obvious advantage of such a technique is independence on a particular form of the medium inhomogeneity. All kinds of possible perturbation, like spatial inhomogeneity of aerosol composition, variation of the single scattering albedo, or phase function characteristics, can be considered and calculated during a single run.

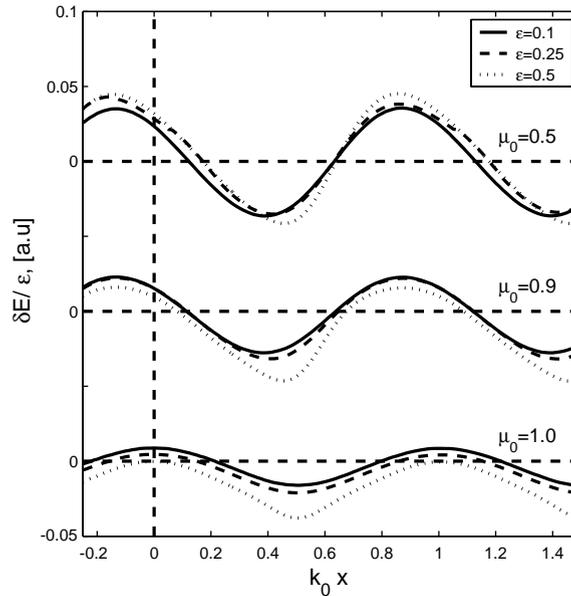


Fig. 7. The ratio $\delta E/\varepsilon$ as a function of $k_0 x$ simulated using the SHDOM. The average cosine of the phase function, g , is 0.0. The wavenumber k_0 is $1.0\bar{\sigma}_e$. The variation strength ε is 0.1 (solid line), 0.25 (dashed line), and 0.5 (dotted line).

The comparison with the SHDOM shows that the developed approach provides high enough accuracy of numerical simulation, especially if the sensitivity to different forms of medium perturbation is under investigation. Another advantage is that our approach is free from the limitations of the diffusion approximation, which has often been used to solve similar problems.

To conclude, we should note that the developed approach allows one to include polarization effects with no great difficulty: a problem which the authors will focus on in the near future.

Acknowledgements

This work is supported by the Australian Research Council. The authors wish to express their special gratitude to Gordon Godfrey Bequest which made our joint work in Sydney during the austral winter of 2001 possible. AD acknowledges partial support from the US Department of Energy's Atmospheric Radiation Measurement Program.

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